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LETTER TO THE EDITOR

Classical diffusion in random fields with long-range correlations

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Abstract. The effect of long-range random static fields on classical diffusion is investigated. It is shown that the long-range component of random fields leads to the appearance of a new (compared to the white noise case) fixed point in the renormalisation group (RG) equations. The critical exponents and oscillating corrections corresponding to the new fixed point are calculated.

In a recent series of papers [1-6] the effect of random static fields on classical diffusion was investigated. The dimension $d = 2$ was shown to be the upper critical dimension for this problem. This means that the $d > 2$ asymptote of the mean-square displacement for long times does not change and the behaviour remains diffusive $\langle \boldsymbol{r}^2(t)\rangle \sim t$. In the case $d = 2 - \varepsilon$ and for an isotropic δ-correlated random field, subdiffusive behaviour was obtained. A universal logarithmic correction to the diffusive behaviour was obtained for $d = 2$ and an isotropic δ-correlated random function. Superdiffusive behaviour at $d = 2(\langle \boldsymbol{r}^2(t)\rangle \sim t(\ln t^{1/2})$ was obtained for transverse anisotropy, and subdiffusive behaviour obtained for longitudinal anisotropy.

The aim of this letter is to investigate the long-range isotropic random field effect on classical diffusion. We show that taking account of the slowly-changing random field leads to the appearance of a new fixed point in the RG equations. At certain parameter values this new fixed point is stable and determines the critical behaviour of the system. In certain cases, the new point represents the stable fixed point of the focus type and therefore gives oscillating corrections to scaling laws.

As usual, we consider the equation of motion for a particle in a viscous medium and in the random static force field

$$\dot{\boldsymbol{r}} = \boldsymbol{\eta}(t) + \boldsymbol{f}(\boldsymbol{r}) \tag{1}$$

where $\boldsymbol{\eta}(t)$ and $\boldsymbol{f}(\boldsymbol{r})$ are Gaussian random functions with correlation functions:

$$\begin{aligned} &\langle \eta_\alpha(t)\eta_\beta(t')\rangle = 2D\delta_{\alpha\beta}\delta(t-t') \\ &\langle f_\alpha(\boldsymbol{r}) f_\beta(\boldsymbol{r}')\rangle = \delta_{\alpha\beta}B(|\boldsymbol{r}-\boldsymbol{r}'|) \\ &\langle \boldsymbol{\eta}\rangle = \langle \boldsymbol{f}\rangle = 0. \end{aligned} \tag{2}$$

The associated Fokker-Planck equation is

$$\frac{\partial P}{\partial t} - D\nabla^2 P + \nabla(\boldsymbol{f}P) = 0. \tag{3}$$

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It is convenient to deal with the Green function of (3) determined by

$$\left(\frac{\partial}{\partial t}-D\nabla^2+\nabla f\right)G(\boldsymbol{r},t,\boldsymbol{r}',t')=\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')$$
$$G(\boldsymbol{r},t,\boldsymbol{r}',t')=0 \qquad t\leqslant t'. \tag{4}$$

Through the Green function G one may obtain all the moments of $\boldsymbol{r}(t)$. We represent G in the form of a functional integral

$$G(\boldsymbol{r},t,\boldsymbol{r}',t')=-\mathrm{i}\int D\psi D\varphi\psi(\boldsymbol{r}',t')\varphi(\boldsymbol{r},t)\exp\left[\mathrm{i}\int\psi\left(\frac{\partial}{\partial t}-D\nabla^2+\nabla\cdot\boldsymbol{f}\right)\varphi\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t\right]. \tag{5}$$

Here $\psi(\boldsymbol{r},t)$ and $\varphi(\boldsymbol{r},t)$ are real functions. Notice that in this time-dependent representation the normalising denominator is absent. This means that all diagrams with closed loops are identically zero (see [3] and [7]). After Gaussian averaging over $\boldsymbol{f}(\boldsymbol{r})$ in (5) we obtain

$$G(\boldsymbol{r},t,\boldsymbol{r}',t')=-\mathrm{i}\int D\psi D\varphi\psi(\boldsymbol{r}',t')\varphi(\boldsymbol{r},t)\exp(\mathscr{L}_0+\mathscr{L}_{\mathrm{int}})$$

where

$$\mathscr{L}_0=\mathrm{i}\int\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}\,\psi(-\omega,-\boldsymbol{q})\varphi(\omega,\boldsymbol{q})(-\mathrm{i}\omega+Dq^2)$$
$$\mathscr{L}_{\mathrm{int}}=-\tfrac{1}{2}\int\mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\varphi(\boldsymbol{r},t)\nabla\psi(\boldsymbol{r},t)B(|\boldsymbol{r}-\boldsymbol{r}'|)\varphi(\boldsymbol{r}',t')\nabla\psi(\boldsymbol{r}',t'). \tag{6}$$

For a δ-correlated random function $B(\boldsymbol{r}-\boldsymbol{r}')=\Delta\delta(\boldsymbol{r}-\boldsymbol{r}')$ we suppose that $B(r)$ at large distances falls off as $1/r^b$. Then the Fourier transform of $B(r)$ at small k has the form

$$B(k)\approx\Delta+Wk^{b-d}. \tag{7}$$

Notice that such a long-range fluctuating field exists for example in the two-dimensional degenerate systems where the field $\boldsymbol{f}(\boldsymbol{r})$ originates from the two-component order parameter. It is known (see e.g. [8]) that in these systems in the low-temperature phase the correlation function at large distances falls off as a power order. One may observe from (7) that the interaction in the Lagrangian is determined by two constants Δ and W, but it is clear that when $b>d$ the parameter W will be irrelevant and all the results for the asymptotic behaviour of $\langle\boldsymbol{r}^2(t)\rangle$ will be the same as in the case of white noise ($W\equiv0$). We will show below that in the case $b<2d-2$ there are essential differences compared with the case of white noise. We use the RG technique in order to find the asymptotic behaviour of $\langle\boldsymbol{r}^2(t)\rangle$ and obtain the RG equations demanding, as usual, that at renormalisation the coefficient q^2 in the Lagrangian does not change:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}l}=\varepsilon\Delta+\frac{W^2-\Delta^2}{4\pi}\qquad\frac{\mathrm{d}W}{\mathrm{d}l}=\delta W-\frac{W(\Delta+W)}{2\pi}$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}l}=z\omega\qquad z=2+\frac{W(\varepsilon-\delta)}{4\pi}+\frac{(\Delta+W)^2}{8\pi^2}. \tag{8}$$

Here $\delta=2-b$, $\varepsilon=2-d$ and also $\delta=\mathrm{O}(\varepsilon)$ and we limit ourselves to the second order of Δ, W, $l = \ln s$ (s is the scale). The system of equations has three fixed points

$$\begin{array}{lll} \text{(i)} & \Delta^* = 0 & W^* = 0 \\ \text{(ii)} & \Delta^* = 4\pi\varepsilon & W^* = 0 \\ \text{(iii)} & \Delta^* = \dfrac{\pi\delta^2}{\delta - \varepsilon} & W^* = \dfrac{\pi\delta^2 - 2\pi\delta\varepsilon}{\delta - \varepsilon}. \end{array} \tag{9}$$

The first point is the Gaussian fixed point, the second point is related to the short-range part of the random field, and the third point is related to the long-range part of the random field (compare with [9]). Now we linearise the system in the vicinity of the fixed points in order to determine stability regions (in the ε, δ plane) of the fixed points. We obtain from (8) assuming $\Delta = \Delta^* + \Delta_1$, $W = W^* + W_1$

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}l} = A\boldsymbol{U} \tag{10}$$

where A is a 2×2 matrix and $\boldsymbol{U}$ is a vector:

$$A = \begin{pmatrix} \varepsilon - \dfrac{\Delta^*}{2\pi} & \dfrac{W^*}{2\pi} \\ -\dfrac{W^*}{2\pi} & \delta - \dfrac{\Delta^*}{2\pi} - \dfrac{W^*}{\pi} \end{pmatrix} \qquad \boldsymbol{U} = \begin{pmatrix} \Delta_1 \\ W_1 \end{pmatrix}. \tag{11}$$

A fixed point is stable when all the eigenvalues of the corresponding matrix A are negative. We find that the ε, δ plane is divided into the following stability regions (see figure 1). In regions 1 and 2, the Gaussian fixed point and short-range fixed point, respectively, are stable. In regions 3, 4 and 5 the long-range fixed point is stable. In region 4 the eigenvalues of A become complex and therefore oscillating corrections to the scaling laws occur.

It is easy to show that at $t \to \infty$

$$\langle \boldsymbol{r}^2(t) \rangle \sim t^{2/z}. \tag{12}$$

Here z is determined by (8), where in each stability region we must insert the corresponding fixed point from (9) instead of Δ and W. Substituting (8) into (12) we obtain the following asymptotics for the mean-square displacement:

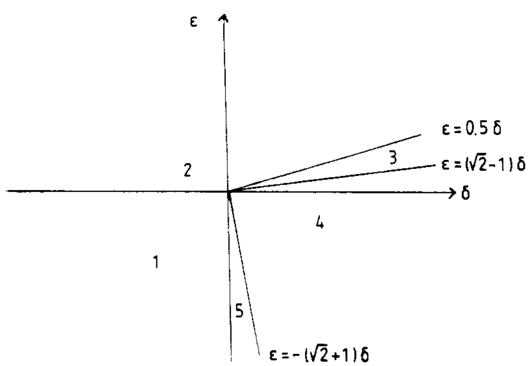

Figure 1. The regions of stability of fixed points.

$$\begin{array}{ll} \text{1st region} & z=2, \langle \boldsymbol{r}^2(t)\rangle \sim t \\ \text{2nd region} & z=2+2\varepsilon^2, \langle \boldsymbol{r}^2(t)\rangle \sim t^{1-\varepsilon^2} \\ \text{3rd–5th regions} & z=2+\frac{1}{4}\delta^2+\frac{1}{2}\delta\varepsilon, \langle \boldsymbol{r}^2\rangle \sim t^{1-\delta^2/8-\delta\varepsilon/4}. \end{array} \tag{13}$$

Notice some peculiarities of (13). It is interesting that when δ increases (or b decreases) the exponent of t decreases (when $\varepsilon \geq 0$). The second peculiarity is the difference from the diffusive behaviour in the case $\varepsilon < 0$ (for example at $d = 3$).

As we have noted above, in region 4 the eigenvalues of matrix A become complex. Therefore in this case the corresponding fixed point is a focus-type stable point in the Δ, W phase plane. In this case oscillating corrections to the scaling laws for $\langle \boldsymbol{r}^2\rangle$ appear, in analogy with the problems considered in [9] and [10]. Although such a correction appears in the whole of region 4, we consider for simplicity the case $\varepsilon = 0 (d = 2)$ and $\delta > 0$. In this case from (8) we have:

$$\ln \omega/\omega_0 = \int_0^l z(l')\,\mathrm{d}l' \tag{14}$$

where

$$z(l) = 2 + \frac{\delta^2}{4} + \frac{\Delta_1(l)\delta}{2\pi} + \frac{\delta W_1(l)}{4\pi}.$$

Here Δ_1 and W_1 are deviations (we limit ourselves to linear approximation on Δ_1 and W_1) from the fixed point $\Delta^* = \pi\delta$, $W^* = \pi\delta$ (see (9) at $\varepsilon = 0$), and satisfy the linear system of differential equations (10), where A is determined by (11) at $\varepsilon = 0$. Solving these equations for Δ_1 and W_1 we obtain

$$\begin{aligned} \Delta_1(l) &= \mathrm{e}^{-\delta l/2}(\Delta_{10} \cos \tfrac{1}{2}\delta l + W_{10} \sin \tfrac{1}{2}\delta l) \\ W_1(l) &= \mathrm{e}^{-\delta l/2}(W_{10} \cos \tfrac{1}{2}\delta l - \Delta_{10} \sin \tfrac{1}{2}\delta l). \end{aligned} \tag{15}$$

Here Δ_{10} and W_{10} are arbitrary constants. The mean-square displacement $\langle \boldsymbol{r}^2\rangle$ tends to e^{2l_0} when $t \to \infty$, where l_0 is the scale at which $\omega(l_0) \sim 1$ if we begin renormalisation from $\omega_0 = 1/t$ at $l = 0$. Substituting (15) into (14) and solving it, we obtain

$$\langle \boldsymbol{r}^2\rangle \sim t^{1-\delta^2/8}\left[1 + t^{-\delta/4}\left(C_1 \cos\frac{\delta \ln t}{4} + C_2 \sin\frac{\delta \ln t}{4}\right)\right] \tag{16}$$

where C_1 and C_2 are arbitrary non-universal constants.

We now assume that $\langle f\rangle = f_0 \neq 0$. Then the Green function in the case $\Delta = W = 0$ has the form:

$$G_0(\omega, \boldsymbol{q}) = \frac{1}{-\mathrm{i}\omega + \mathrm{i}f_0\boldsymbol{q} + q^2}. \tag{17}$$

In this case the diffusion coefficient is determined as

$$\langle X_{\mu\nu}(t)\rangle = 2D_{\mu\nu}(t)t \tag{18}$$

where

$$\langle X_{\mu\nu}(t)\rangle = \langle X_\mu X_\nu\rangle - \langle X_\mu\rangle\langle X_\nu\rangle.$$

If $\Delta = W = 0$ from (17) we obtain

$$D^0_{\mu\nu}(t) = \delta_{\mu\nu}. \tag{19}$$

We analyse the behaviour of $D_{\mu\nu}(t)$ at $t\to\infty$ and $f_0\to 0$. Notice that all the results above are correct at $t \ll f_0^{-2}$. In the crossover region $t \sim t_c \sim f_0^{-2}$ the behaviour of $D_{\mu\nu}(t)$ is changed. In certain cases (see below and also [5]) corrections to exponent 2 in the crossover region appear. The f_0 as ω plays the role of a cut-off factor in the logarithmic divergent integrals.

The renormalisation of f_0 is determined by the equation

$$\frac{\mathrm{d}f_0}{\mathrm{d}l} = f_0\left(1 - \frac{\Delta + W}{8\pi}\right). \tag{20}$$

On the large scales l (small times) the perturbation theory is applicable, and we obtain

$$\langle X_{\mu\nu}[t(l_0)]\rangle \sim 2\delta_{\mu\nu} t(l_0) \sim \mathrm{e}^{-2l_0}\langle X_{\mu\nu}(t)\rangle \sim \mathrm{e}^{-2l_0} 2D_{\mu\nu}(t)t. \tag{21}$$

It follows from (20) and (21) that

$$\begin{aligned} &D_{\mu\nu}(t) \sim D\delta_{\mu\nu} \\ &D \sim \mathrm{e}^{2l_0} \exp\left(-\int_0^l z(l')\,\mathrm{d}l'\right). \end{aligned} \tag{22}$$

Substituting (8) into (22) we find

$$D \sim \exp\left[-l_0\left(\frac{W^*(\varepsilon-\delta)}{4\pi} + \frac{(\Delta^* + W^*)^2}{8\pi^2}\right)\right]. \tag{23}$$

In the case $t \ll t_c$ the divergent integrals are cut off by the frequency and, therefore, the scale l_0 is determined from the condition

$$t(l_0) \sim 1/\omega(l_0) \sim t \exp\left(-\int_0^{l_0} z(l)\,\mathrm{d}l\right) \sim 1.$$

At $t \gg t_c$ the integrals are cut off by the average force f_0 and therefore we find l_0 from the condition $f_0(l) \sim 1$. The crossover takes place when

$$f_0^2(l) \sim t^{-1}(l). \tag{24}$$

We find the crossover time from (20) and (24)

$$t_c \sim f_0^{-2-(W^*+\Delta^*)/4\pi}.$$

In each stability region we must insert the corresponding fixed point (see above) instead of Δ^* and W^*. For example, in regions 3-5 we have $t_c \sim f_0^{-2-\delta/2}$. Using (22) we obtain the diffusion coefficient when $t \gg t_c$ and $f_0 \to 0$. In regions 3-5 it has the form

$$D \sim f_0^{\delta\varepsilon/2+\delta^2/4}. \tag{25}$$

In regions 1 and 2 D takes the values $D \sim 1$, $D \sim f_0^{2\varepsilon^2}$ respectively [2, 5].

The oscillating correction to the scaling law (25) originates in region 4 because the long-range fixed point is of focus type. Consider the case $d = 2(\varepsilon = 0)$, for simplicity. In this case substituting (15) into (22) and using $l_0 = -\ln f_0$ in the region $t \gg t_c$ we have

$$D \sim f_0^{\delta^2/4}\left(1 + C_1 f_0^{\delta/2} \sin\frac{\delta \ln f_0}{2} + C_2 f_0^{\delta/2} \cos\frac{\delta \ln f_0}{2}\right). \tag{26}$$

Here C_1 and C_2 are non-universal arbitrary constants.

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